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Topics on Variational Analysis and Applications to Equilibrium Problems

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Abstract. We observe how many equilibrium problems obey a generalized complementarity condition, which in general leads to a variational inequality. We illustrate this fact, by studying the elastic–plastic torsion problem and finding the related Lagrange multipliers.

Key words: Complementary conditions, Elastic–plastic torsion problem, Equilibrium problems, Lagrangean theory, Quasi–relative interior, Variational inequalities

1. Introduction: What does Optimality means?

The study of many equilibrium problems (The Obstacle Problem, the discrete traffic equilibrium problem, the continuous traffic equilibrium problem, the spatial price equilibrium problem, the migration problem, the Walras problem, etc. (see [8], [9], [14])) has contributed to focus the fact that the equilibrium conditions obey a form of generalized complementarity conditions whose meaning is that when one of the factors is greater than zero, the other one must be zero.

Then one obtains the remarkable fact that different problems in different context obey a unique law, which, in general, is far to correspond to the usual “obedience” to be the minimum of a functional.

An open problem remains to see: how this set of generalized complementarity conditions can be successfully treated? However, until now, each of the problems above mentioned can be transformed in a Variational Inequality on a convex subset $\mathbb{K}$ of a suitable functional space, for which we have an impressive quantity of results in terms of existence, calculation of the solutions, stability and sensitivity analysis.

Now the aim of this paper is to show how the possibility of transforming the complementarity conditions in terms of Variational Inequality also happens for the “Elastic-Plastic Torsion Problem”. The Lagrangean theory in the infinite dimensional case plays an extraordinary role in order to obtain the proof of the above claims.

In the infinite dimensional context, following the suggestions of Borwein-Lewis (see [1]), the introduction of a generalized Slater condition, that replaces the usual one in the finite dimensional case, solves the delicate problem to have some kind of “interior” of a set non-empty.

2. The Elastic-Plastic torsion model

We can say that the elastic–plastic torsion problem can be described in the following way (Von Mises, 1949, as reported by Ting see [15]):

> Find a function $u(x)$ which vanishes on $\partial\Omega$ and, together with its first derivatives, is continuous in Ω. The gradient of u must not have an absolute value greater than a given constant k on $\overline{\Omega}$. Whenever the absolute value of ∇u is smaller than k, the function u must satisfy the differential equation
>
> $$\Delta u = -2\mu\vartheta,$$
>
> where the positive constants μ and ϑ denote the shearing modulus and the angle of twist per unit length, respectively.

3. Variational Inequality Formulation of the Elastic-Plastic Torsion Problem

Let Ω be an open bounded Lipschitz domain with its boundary $\partial\Omega$. Let $\mathbb{K}$ be the closed convex non empty subset of $H_0^1(\Omega)$:

$$\mathbb{K} = \left\{ v \in H_0^1(\Omega) : v \geqslant 0, \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2 \leqslant 1 \text{ a.e. in } \Omega \right\}. \tag{1}$$

Let $a(u, v)$ be the bilinear form:

$$a(u,v) = \int_\Omega \left\{ \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} v + \sum_{i=1}^{n} c_i u \frac{\partial v}{\partial x_i} + \mathrm{d} u v + a_0 u v \right\} \mathrm{d}x \tag{2}$$

with

$$\begin{aligned}
& a_{ij} \in L^\infty(\Omega),\ b_i \in L^n(\Omega),\ c_i \in L^n(\Omega) \quad \text{if } n > 2 \\
& b_i,\ c_i \in L^{2+\varepsilon}(\Omega),\ \varepsilon > 0 \quad \text{if } n = 2 \\
& d \in L^{n+2}(\Omega) \quad \text{if } n > 2,\ d \in L^{2+\varepsilon}(\Omega),\ \varepsilon > 0 \quad \text{if } n > 2 \\
& \sum_{ij=1}^{n} a_{ij} \xi_i \xi_j \geqslant \overline{\nu} \sum_{i=1}^{n} \xi_i^2 \quad \forall \xi \in \mathbb{R}^n, \\
& a_0 > 0 \quad \text{constant such that } a(u,u) \geqslant \nu \|u\|^2_{H_0^1(\Omega)}, \quad \nu > 0.
\end{aligned} \tag{3}$$

Then, for each $f \in L^2(\Omega)$, the Variational Inequality:

$$\text{“Find } u \in \mathbb{K} \text{ such that } a(u, v-u) \geqslant \int_\Omega f(v-u)\,\mathrm{d}x \quad \forall v \in \mathbb{K}\text{"} \tag{4}$$

admits a unique solution. Moreover, under additional assumptions on the coefficients a_{ij}, e.g., $a_{ij} \in C^1(\overline{\Omega})$ and on b_i, c_i, d, e.g., b_i, c_i, $d \in L^\infty(\Omega)$, it is possible to show that, if $f(x) \in L^p(\Omega)$, $p \geqslant 2$, the solution u to the Variational Inequality (4) belongs to $W^{2,p}(\Omega)$ and that, denoting by $\mathcal{L}u$ the operator such that

$$\langle \mathcal{L}u, v \rangle = a(u, v) - \int_\Omega f(x) v(x)\, dx, \tag{5}$$

it results:

$$\mathcal{L}u \in L^p(\Omega)$$

(see [2]).

Then we can prove the following theorem.

THEOREM 1. Let u be the solution to the Variational Inequality (4). Then there exist $\overline{\mu}$, $\overline{\lambda} \in L^2(\Omega)$ such that a. e. in Ω

$$\overline{\mu}(x)u(x) = 0, \quad \overline{\lambda}(x)\left(1 - \sum_{i=1}^n \left(\frac{\partial u(x)}{\partial x_i}\right)^2\right) = 0,$$
$$\mathcal{L}u(x) - \overline{\mu}(x) = 2\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\overline{\lambda}(x)\frac{\partial u(x)}{\partial x_i}\right).$$

Moreover, it results:

$$\begin{cases} (\mathcal{L}u(x) - \overline{\mu}(x))\left(1 - \sum_{i=1}^n \left(\dfrac{\partial u(x)}{\partial x_i}\right)^2\right) = 0 \\ u(x) \geqslant 0 \\ 1 - \sum_{i=1}^n \left(\dfrac{\partial u(x)}{\partial x_i}\right)^2 \geqslant 0. \end{cases}$$

The above result can be compared with the results by [14] and [13], whereas equivalence results with respect to the Variational Inequality depending on $\delta(x) = d(x, \partial\Omega)$ can be found in [4, 6, 7, 10–12].

4. Proof of theorems

Let u be the unique solution to the Variational Inequality (4) and let us consider the Lagrangean function:

$$\mathcal{L}(v,\mu,\lambda)=\psi(v)-\int_{\Omega}\mu(x)v(x)\,\mathrm{d}x-\int_{\Omega}\lambda(x)\left(1-\sum_{i=1}^{n}\left(\frac{\partial u(x)}{\partial x_i}\right)^2\right)\mathrm{d}x \tag{6}$$

where

$$\psi(v)=\langle\mathcal{L}u,v-u\rangle \quad v\in H_0^1(\Omega),$$

$$(\mu,\lambda)\in\mathcal{C}^*=\left\{(\mu,\lambda):\mu\in L^2(\Omega),\ \lambda\in L^2(\Omega),\ \mu(x),\lambda(x)\geqslant 0 \text{ a. e. in } \Omega\right\}. \tag{7}$$

Now let us observe that the set:

$$C=\{v\in L^p(\Omega): v(x)\geqslant 0 \text{ a. e. in } \Omega\}$$

has an empty interior; so that the Slater constraint qualification cannot be applied.

Following a suggestion by Borwein–Lewis (see [1]), it is possible to overcome this difficulty replacing the Slater qualification condition by generalizing the notion of relative interior as follows:

DEFINITION 1. The quasi relative interior of a convex set C, which we denote by qri C, is the set of those x for which

$$\text{Cl Cone } (C-x) \tag{8}$$

is a subspace.

We have:

$$\text{Cone } (C-x)=\{\lambda y:\lambda\geqslant 0,\ y\in C-x\}$$

and if C is starshaped with respect to x, it results

$$T(C,x)=\text{ Cl Cone } (C-x).$$

Then the generalized condition is

$$\text{qri } C\neq\emptyset \tag{9}$$

and it results

$$\text{qri } \mathcal{C}^*=\left\{(\lambda,\mu)\in L^2(\Omega):\lambda(x)>0,\ \mu(x)>0\right\},$$

$$\text{qri } \mathbb{K}\neq\emptyset.$$

Then the usual Lagrangean theory can be applied and we obtain this lemma (see for instance [6]):

LEMMA 1. *There exists* $(\overline{\mu},\overline{\lambda})\in\mathcal{C}^*$ *such that*

$$\mathcal{L}(u,\mu,\lambda)\leqslant\mathcal{L}(u,\overline{\mu},\overline{\lambda})\quad\forall v\in H_0^1(\Omega)\tag{10}$$

$$\mathcal{L}(v,\overline{\mu},\overline{\lambda})\geqslant\mathcal{L}(u,\overline{\mu},\overline{\lambda})\quad\forall(\lambda,\mu)\in\mathcal{C}^*\tag{11}$$

and $\mathcal{L}(u,\overline{\mu},\overline{\lambda})=0$, *i. e.*

$$\psi(u)=0,\ \int_\Omega u(x)\overline{\mu}(x)\,\mathrm{d}x=0,\ \int_\Omega\overline{\lambda}(x)\left(1-\sum_{i=1}^n\left(\frac{\partial u(x)}{\partial x_i}\right)^2\right)\mathrm{d}x=0.$$

Being

$$1-\sum_{i=1}^n\left(\frac{\partial u(x)}{\partial x_i}\right)^2\geqslant 0.$$

$$\overline{\mu}(x)\geqslant 0,\quad\overline{\lambda}(x)\geqslant 0,\quad u(x)\geqslant 0,$$

we derive

$$\overline{\mu}(x)u(x)=0\text{ a.e. in }\Omega,\tag{12}$$

$$\overline{\lambda}(x)\left(1-\sum_{i=1}^n\left(\frac{\partial u(x)}{\partial x_i}\right)^2\right)=0\text{ a.e. in }\Omega\tag{13}$$

$$\text{i.e.}\quad\overline{\lambda}(x)=\overline{\lambda}(x)\sum_{i=1}^n\left(\frac{\partial u(x)}{\partial x_i}\right)^2\text{ a.e. in }\Omega.\tag{14}$$

From (11), (12) and (14) we get

$$\begin{aligned}\mathcal{L}(v,\overline{\mu},\overline{\lambda}) &= \langle\mathcal{L}u,v-u\rangle-\int_\Omega\overline{\mu}(x)(v(x)-u(x))\,\mathrm{d}x\\ &\quad-\int_\Omega\sum_{i=1}^n\overline{\lambda}(x)\left[\left(\frac{\partial u(x)}{\partial x_i}\right)^2-\left(\frac{\partial v(x)}{\partial x_i}\right)^2\right]\mathrm{d}x\\ &=\langle\mathcal{L}u,v-u\rangle-\int_\Omega\overline{\mu}(x)(v(x)-u(x))\,\mathrm{d}x\\ &\quad-\int_\Omega\sum_{i=1}^n\overline{\lambda}(x)\left(\frac{\partial u(x)}{\partial x_i}-\frac{\partial v(x)}{\partial x_i}\right)\left(\frac{\partial u(x)}{\partial x_i}+\frac{\partial v(x)}{\partial x_i}\right)\mathrm{d}x\\ &=\langle\mathcal{L}u,v-u\rangle-\int_\Omega\overline{\mu}(x)(v(x)-u(x))\,\mathrm{d}x\end{aligned}$$

$$-\left\langle\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\overline{\lambda}\frac{\partial u}{\partial x_i}\right), v-u\right\rangle-\left\langle\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\overline{\lambda}\frac{\partial v}{\partial x_i}\right), v-u\right\rangle\geqslant 0$$
$$\forall v\in H_0^1(\Omega). \tag{15}$$

Choosing $v=u\pm\psi\ \forall\psi\in H_0^1(\Omega)$, we get

$$\langle \mathcal{L}u,\psi\rangle-\langle\overline{\mu},\psi\rangle-2\left\langle\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\overline{\lambda}\frac{\partial u}{\partial x_i}\right),\psi\right\rangle-\left\langle\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\overline{\lambda}\frac{\partial \psi}{\partial x_i}\right),\psi\right\rangle\geqslant 0$$

and

$$\langle \mathcal{L}u,\psi\rangle-\langle\overline{\mu},\psi\rangle-2\left\langle\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\overline{\lambda}\frac{\partial u}{\partial x_i}\right),\psi\right\rangle+\left\langle\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\overline{\lambda}\frac{\partial \psi}{\partial x_i}\right),\psi\right\rangle\leqslant 0.$$

Then considering the test functions $\varepsilon\,\psi$, $\varepsilon>0$ in both the inequalities and letting ε tend to zero, we get

$$\left\langle\mathcal{L}u-\overline{\mu}-2\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\overline{\lambda}\frac{\partial u}{\partial x_i}\right),\psi\right\rangle\geqslant 0\quad\forall\psi\in H_0^1(\Omega)$$
$$\left\langle\mathcal{L}u-\overline{\mu}-2\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\overline{\lambda}\frac{\partial u}{\partial x_i}\right),\psi\right\rangle\leqslant 0\quad\forall\psi\in H_0^1(\Omega)$$

and hence

$$\mathcal{L}u(x)-\overline{\mu}(x)-2\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\overline{\lambda}(x)\frac{\partial u(x)}{\partial x_i}\right)=0\text{ a.e. in }\Omega. \tag{16}$$

Then the first part of theorem is proved (note that the term $\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\overline{\lambda}\frac{\partial u}{\partial x_i}\right)$ belongs to $L^2(\Omega)$).

Now if we consider $x\in E$ where

$$E=\left\{x\in\Omega:\sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2<1\right\},$$

from (13) we deduce, taking into account (16)

$$\mathcal{L}u(x)-\overline{\mu}(x)=0. \tag{17}$$

On the other hand, if $\mathcal{L}u(x)-\overline{\mu}(x)\neq 0$, also

$$2\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\overline{\lambda}(x)\frac{\partial u(x)}{\partial x_i}\right)\neq 0$$

and $\overline{\lambda}(x)$ cannot be zero; hence in the same region it must be

$$x \in P = \left\{ x \in \Omega : \sum_{i=1}^{n} \left(\frac{\partial u(x)}{\partial x_i} \right)^2 = 1 \right\},$$

and we find that if u is the solution to the Variational Inequality (4), it results a.e. in Ω:

$$\begin{cases} (\mathcal{L}u(x) - \overline{\mu}(x)) \left(1 - \sum_{i=1}^{n} \left(\frac{\partial u(x)}{\partial x_i} \right)^2 \right) = 0 \\ u(x) \geqslant 0 \\ 1 - \sum_{i=1}^{n} \left(\frac{\partial u(x)}{\partial x_i} \right)^2 \geqslant 1. \end{cases}$$

REMARK 1. It is easy to show that if $u \in \mathbb{K}$ and there exist $\overline{\lambda}$ and $\overline{\mu}$ as in theorem 1, then u verifies Variational Inequality (4).

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